Negative-energy perturbations in cylindrical equilibria with a radial electric field

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The impact of an equilibrium radial electric field **E** on negative-energy perturbations (NEP's) in cylindrical equilibria of magnetically confined plasmas is investigated within the framework of Maxwell-drift kinetic theory. It turns out that for wave vectors with a nonvanishing component parallel to the magnetic field, the conditions for the existence of NEP's in equilibria with $\mathbf{E}=\mathbf{0}$ [G. N. Throumoulopoulos and D. Pfirsch, Phys. Rev. E **53**, 2767 (1996)] remain valid, while the condition for the existence of perpendicular NEP's, which are found to be the most important perturbations, is modified. For $|e_i\phi|\approx T_i$, a scaling which is satisfied in the edge region of magnetic confinement systems (ϕ is the electrostatic potential), the impact of **E** on perpendicular NEP's depends on the value of T_i/T_e, i.e., (a) for $T_i/T_e<\beta_c\approx P/(B^2/8\pi)$ (P is the total plasma pressure) the electric field does not have any effect; and (b) for $T_i/T_e>\beta_c$, a case which is of operational interest in magnetic confinement systems, the existence of perpendicular NEP's depends on $e_\nu\mathbf{E}$, where e_ν is the charge of the particle species ν. In the latter case, for tokamaklike equilibria and H mode parameters pertaining to the plasma edge two regimes of NEP's exist. In the one of them the critical value $\frac{2}{3}$ of $\eta_i\equiv\partial\ \ln T_i/\partial\ln N_i$ plays a role in the existence of ion NEP's, as in equilibria with $\mathbf{E}=\mathbf{0}$, while a critical value of η_e does not occur for the existence of electron NEP's. However, **E** has a ''stabilizing'' effect on both particle species in that (a) the portion of particles associated with NEP's (active particles) is nearly independent of the plasma magnetic properties, i.e., it is nearly the same in a diamagnetic plasma and in a paramagnetic plasma, while in equilibria with $\mathbf{E}=\mathbf{0}$ this portion is much larger in a paramagnetic plasma than in a diamagnetic plasma; and (b) the fraction of active particles can decrease from the plasma interior to the edge, e.g., for the case of electron NEP's in an equilibrium of a diamagnetic plasma, contrary to equilibria with $\mathbf{E}=\mathbf{0}$. In particular, the fraction of active electrons decreases with increasing η_e and for $\eta_e>\eta_0\approx\frac{4}{3}$ the electric field stabilizes the electrons, in that the fraction of active electrons becomes smaller than the one corresponding to equilibria with $\mathbf{E}=\mathbf{0}$. In addition, **E** has similar stabilizing effects on electron NEP's in stellaratorlike equilibria with pressure profiles identical to those of tokamaklike equilibria, while it results in an increase of the fraction of active ions in reversed-field-pinchlike equilibria. The present results indicate that the radial electric field reduces the NEP's activity in the edge region of tokamaks and stellarators, the reduction of electron NEP's being more pronounced than that of ion NEP's. [S1063-651X(97)11811-6]

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I. INTRODUCTION

Negative-energy waves are potentially dangerous because they can lead to either linear instability [1] or nonlinear, explosive instability [2–20]. Expressions for the second variation of the free energy $F^{(2)}$ were derived by Pfirsch and Morrison [7] for arbitrary perturbations of general equilibria within the framework of dissipationless Maxwell-Vlasov and drift kinetic theories. It was also found that negative-energy perturbations exist in any Maxwell-Vlasov equilibrium whenever the unperturbed distribution function $f_\nu^{(0)}$ of any particle species ν deviates from monotonicity and/or isotropy in the vicinity of a single point, i.e., whenever the condition

$$(\mathbf{k}\cdot\mathbf{v})\left(\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right)>0 \tag{1}$$

holds (in the frame of reference of minimum equilibrium energy) for any particle species ν for some position vector **x** and velocity **v** and for some local vector **k**. The proof of this result was based on infinitely strongly localized perturbations, which correspond to $|\mathbf{k}|\rightarrow\infty$. This raises the question of the degree of localization actually required for negative-energy perturbations (NEP's) to exist in a certain equilibrium. Studying a homogeneous Maxwell-Vlasov plasma [8], force-free equilibria with a sheared magnetic field [9] and general one- and two-dimensional equilibria of magnetically confined plasmas [10–12], Correa-Restrepo and Pfirsch showed that NEP's exist for any deviation of the equilibrium distribution function of any of the species from monotonicity and/or isotropy, without having to impose any restricting conditions on **k**.

NEP's which are not strongly localized can be investigated more conveniently in the framework of Maxwell-drift kinetic theory, which eliminates from the outset all perturbations with perpendicular wavelengths smaller than the gyroradius. In the context of this theory, for a homogeneous magnetized plasma it was found that NEP's exist for any wave vector **k** with a nonvanishing component parallel to the magnetic field (parallel and oblique modes) whenever the condition

$$v_\parallel\frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel}>0 \tag{2}$$

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is satisfied for the equilibrium guiding center distribution function $f_{g\nu}^{(0)}$ for some particle species ν and parallel velocity $v_\parallel$ in the frame of lowest equilibrium energy [7]. For the more interesting cases of inhomogeneous magnetically confined plasmas and equilibria depending on just one Cartesian coordinate x [17] and cylindrical equilibria with vanishing electric fields [18,19], in addition to parallel and oblique modes for which condition (1) also applies, perpendicular NEP's are possible. The latter are the most important perturbations because they can exist even if $v_\parallel(\partial f_{g\nu}^{(0)}/\partial v_\parallel)<0$, which is satisfied, e.g., for Maxwellian distribution functions for all $v_\parallel$. In plane geometry the pertinent condition is

$$\frac{dP^{(0)}}{dx}\frac{\partial f_{g\nu}^{(0)}}{\partial x}<0,$$

where $P^{(0)}$ is the equilibrium plasma pressure. For tokamak-like equilibria with singly peaked pressure profiles the existence of both ion and electron perpendicular NEP's is associated with the critical value $\frac{2}{3}$ of the quantity $\eta_\nu \equiv \partial \ln T_\nu/\partial \ln N_\nu$ (T_ν is the temperature and N_ν the density of particle species ν) which usually governs the onset of the temperature gradient driven modes. For cylindrical equilibria an additional regime of NEP's exists, related to the curvature of the poloidal magnetic field. Also, for the case of cold-ion equilibria ($T_i=0$) a large portion of electrons is associated with NEP's (active particles).

The purpose of the present paper is twofold: (a) to investigate the impact of a radial electric field on NEP's in cylindrical equilibria of magnetically confined plasmas, and (b) to extend the study to equilibria with $T_i \neq 0$. The method of investigation consists in evaluating the general expression $F^{(2)}$ for the second-order perturbation energy within the framework of the linearized dissipationless Maxwell-drift kinetic theory. This is the subject of Sec. II. The conditions for the existence of NEP's are obtained in Sec. III. It turns out that for parallel and oblique perturbations condition (2) remains valid, while the condition for perpendicular NEP's (which remain the most important perturbations) is modified. To apply the condition for perpendicular NEP's in equilibria of magnetic confinement systems the equilibrium equations are needed, which are derived in Sec. IV. Shearless stellaratorlike equilibria are possible with local Maxwellian distribution functions, while tokamaklike and reversed-field-pinchlike equilibria can be obtained from shifted Maxwellian distribution functions, which imply net toroidal currents. For these kinds of distribution functions and H mode parameters pertaining to the plasma edge the condition for the existence of perpendicular NEP's is applied in Sec. V, and the effect of $\mathbf{E}$ on the threshold value of η_ν is examined. In Sec. VI the impact of $\mathbf{E}$ on the fraction of active particles is investigated for shearless stellaratorlike, tokamaklike, and reversed-field-pinchlike analytic equilibria. Our conclusions are summarized in Sec. VII.

II. EQUILIBRIUM AND SECOND-ORDER PERTURBATION ENERGY

We start with a brief outline of the Maxwell-drift kinetic theory adapted to the needs of the present study. More details can be found in Ref. [7].

The expression for the free energy $F^{(2)}$ upon arbitrary perturbations of general equilibria is given by

$$F^{(2)}=\int d^3x\; T_0^{(2)0}, \tag{3}$$

where $T_0^{(2)0}$ is the energy component of the second-order energy-momentum tensor $T_\rho^{(2)\lambda}$. To derive the tensor $T_\rho^{(2)\lambda}$, Pfirsch and Morrison [7] used the following modified Hamilton-Jacobi formalism. Let $H_\nu(p_i,q^i,t)$ be the Hamiltonian for particles of species ν for the perturbed state in a phase space $p_1,\ldots,p_4$, $q^1,\ldots,q^4$, where (q^1,q^2,q^3) are generalized coordinates so that $\mathbf{x}=\mathbf{x}(q^1,q^2,q^3)$ and correspondingly $\mathbf{p}=\mathbf{p}(p_1,p_2,p_3)$, where $\mathbf{x}$ is the position vector in normal space; p_4, q^4 is an additional pair of canonical variables which is needed to describe guiding center motion. Let $H_\nu^{(0)}(P_i,Q^i)$ be the equilibrium Hamiltonian in the phase space $P_1,\ldots,P_4$, $Q^1,\ldots Q^4$, and let $S_\nu(P_i,q^i,t)$ be a mixed-variable generating function for a canonical transformation between p_i, q^i and P_i, Q^i. The $\mathbf{x}$, t dependence of H_ν is given via electromagnetic potentials $\phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$, the electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ and their derivatives. The quantities p_i and Q^i are obtained from S_ν as

$$p_i=\frac{\partial S_\nu}{\partial q^i},\quad Q^i=\frac{\partial S_\nu}{\partial P_i}, \tag{4}$$

and S_ν must be the solution of the modified Hamilton-Jacobi equation

$$\frac{\partial S_\nu}{\partial t}+H_\nu\left(\frac{\partial S_\nu}{\partial q^i},q^i,t\right)=H_\nu^{(0)}\left(P_i,\frac{\partial S_\nu}{\partial P_i}\right). \tag{5}$$

The time-independent, zeroth-order solution $S_\nu^{(0)}$ of Eq. (5), needed to obtain $T_\rho^{(2)\mu}$, is then simply given by the identity transformation $S_\nu^{(0)}=\Sigma_\nu P_i q^i$.

The theory can be derived from the Lagrangian

$$L=-\sum_\nu \int dq\; dP\; \varphi_\nu(P_i,q^i,t)\left[\frac{\partial S_\nu}{\partial t}+H_\nu\left(\frac{\partial S_\nu}{\partial q^i},q^i,t\right)\right.$$
$$\left.-H_\nu^{(0)}\left(P_i,\frac{\partial S_\nu}{\partial P_i}\right)\right]-\frac{1}{8\pi}\int d^3x(\mathbf{E}^2-\mathbf{B}^2). \tag{6}$$

Here, $dq\; dP\equiv dq^1\ldots dq^4 dP_1\ldots dP_4$; φ_ν are density functions related to the *particle* distribution functions f_ν, and the latter are related to the guiding center distribution functions $f_{g\nu}$ by Eq. (14) below. The energy-momentum tensor can be obtained by using the Euler-Lagrange equations resulting from the variational principle

$$\delta\int_{t_1}^{t_2} L\; dt=0, \tag{7}$$

(with φ_ν, S_ν, ϕ, and $\mathbf{A}$ the quantities to be varied) and Noether's theorem. In the context of the linearized theory, one obtains

$$T_\rho^{(2)\lambda} = -\sum_\nu \int d\tilde{\bar{q}}\, d\tilde{P}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}^\rho} - \frac{e_\nu}{c} A_\rho^{(1)}\right)\left[f_\nu^{(0)}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}^\kappa} - \frac{e_\nu}{c} A_\kappa^{(1)}\right) \frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda \partial \tilde{P}_\kappa} + f_\nu^{(0)} F_{\tau\sigma}^{(1)} \frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda \partial F_{\tau\sigma}^{(0)}} + \left(f_\nu^{(0)} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\right)_{,i} \frac{\partial \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda}\right] - 2F_{\mu\rho}^{(1)} \sum_\nu \int d\tilde{\bar{q}}\, d\tilde{P} \times \left[f_\nu^{(0)}\left(\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}^\kappa} - \frac{e_\nu}{c} A_\kappa^{(1)}\right) \frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_\kappa \partial F_{\mu\lambda}^{(0)}} + f_\nu^{(0)} F_{\sigma\tau}^{(1)} \frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)} \partial F_{\sigma\tau}^{(0)}}\right] - \frac{1}{4\pi} F_{\mu\rho}^{(1)} F^{(1)\mu\lambda} + \delta_\rho^\lambda \left(\sum_\nu \int d\tilde{\bar{q}}\, d\tilde{P}\, f_\nu^{(0)} (\mathcal{H}_\nu^{(2)} - \mathcal{H}_\nu^{(0)(2)}) + \frac{1}{16\pi} F_{\tau\sigma}^{(1)} F^{(1)\tau\sigma}\right). \tag{8}$$

Here, the superscripts (0), (1), and (2) denote equilibrium, first-order, and second-order quantities; the tilde signifies that the time is included, i.e.,

$$(\tilde{q}^{\,i}) = (\tilde{q}^{\,0},\dots,\tilde{q}^{\,4}) = (ct,\mathbf{x},q^4),$$

$$(\tilde{p}_i) = (\tilde{p}_0,\dots,\tilde{p}_4) = (p_0,\mathbf{p},p_4), \quad cp_0 = \frac{\partial S_\nu}{\partial t},$$

$$(\tilde{Q}^i) = (\tilde{Q}^0,\dots,\tilde{Q}^4) = (ct,\mathbf{x},Q^4),$$

$$(\tilde{P}_i) = (\tilde{P}_0,\dots,\tilde{P}_4) = (P_0,\mathbf{p},p_4), \quad P_0 = \text{const},$$

$$\mathcal{H}_\nu(\tilde{p}_i,\tilde{q}^{\,i}) = cp_0 + H_\nu(p_1,\dots,p_4,q^1,\dots,q^4),$$

$$\mathcal{H}_\nu^{(0)}(\tilde{P}_i,\tilde{Q}^i) = cP_0 + H_\nu(P_1,\dots,P_4,Q^1,\dots,Q^4);$$

$d\tilde{\bar{q}}\, d\tilde{P} = dq^4 dP_1 \dots dP_4$; $A_\mu = (-\phi,\mathbf{A})$ with $A_4 \equiv 0$; $F_{\mu\nu}$ is the electromagnetic tensor; the symbol $C^i_{,j}$ signifies covariant derivative of the vector $\mathbf{C}$ with contravariant components C^i:

$$C^i_{,j} \equiv \frac{\partial C^i}{\partial q^j} - \Gamma^i_{jl} C^l,$$

where Γ^i_{jl} are the Christoffel symbols; the scalar quantity $[f_\nu^{(0)}(\partial S_\nu^{(1)}/\partial \tilde{P}_i)]_{,i}$, which replaces $(\partial/\partial \tilde{q}_i)[f_\nu^{(0)}(\partial S_\nu^{(1)}/\partial \tilde{P}_i)]$ in Eq. (46) of Ref. [7], results from the contraction in the tensor $[f_\nu^{(0)}(\partial S_\nu^{(1)}/\partial \tilde{P}_i)]_{,j}$.

The Hamiltonians H_ν for the guiding center motion of a particle species ν, which appear in $F^{(2)}$, are obtained from Littlejohn's Lagrangian formulation of the guiding center theory [21] in the form given by Wimmel [22]:

$$L_\nu = \left(\frac{e_\nu}{c}\right) \mathbf{A}_\nu^\star \cdot \dot{\mathbf{x}} - e_\nu \phi_\nu^\star, \tag{9}$$

with

$$\mathbf{A}_\nu^\star = \mathbf{A} + \frac{m_\nu c}{e_\nu} q^4 \mathbf{b},$$

$$e_\nu \phi_\nu^\star = e_\nu \phi + \mu B + \frac{m_\nu}{2}[(q^4)^2 + \mathbf{v}_E^2],$$

$$\mathbf{v}_E = c\, \frac{\mathbf{E}\times\mathbf{B}}{B^2},$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla\times\mathbf{A}, \quad \mathbf{b} = \frac{\mathbf{B}}{B}.$$

The Euler-Lagrange equations yield $q^4 = \mathbf{v}\cdot\mathbf{b} = v_\parallel$, and the guiding center velocity $\dot{\mathbf{x}} = \mathbf{v} \equiv \mathbf{v}_g$ and $\dot{q}^4$ as functions of t, $\mathbf{x}$, and q^4:

$$\dot{\mathbf{x}} = \mathbf{v} = \mathbf{v}_{g\nu}(t,\mathbf{x},q^4) = \frac{q^4}{B_{\nu\parallel}^\star}\mathbf{B}_\nu^\star + \frac{c}{B_{\nu\parallel}^\star}\mathbf{E}_\nu^\star \times \mathbf{b} \tag{10}$$

and

$$\dot{q}^4 = V^4(t,\mathbf{x},q^4) = \frac{e_\nu}{m_\nu}\frac{1}{B_{\nu\parallel}^\star}\mathbf{E}_\nu^\star\cdot\mathbf{B}_\nu^\star. \tag{11}$$

Here, $\mathbf{E}_\nu^\star \equiv -\nabla\phi_\nu^\star - (1/c)(\partial \mathbf{A}_\nu^\star/\partial t)$, $\mathbf{B}_\nu^\star \equiv \nabla\times\mathbf{A}_\nu^\star$, and $B_{\nu\parallel}^\star \equiv \mathbf{B}_\nu^\star\cdot\mathbf{b}$. The momenta canonically conjugated to $\mathbf{x}$ and q^4 follow from Eq. (9):

$$\mathbf{p} = \frac{\partial L_\nu}{\partial \dot{\mathbf{x}}} = \frac{e_\nu}{c}\mathbf{A}_\nu^\star, \quad p_4 = \frac{\partial L_\nu}{\partial \dot{q}^4} = 0. \tag{12}$$

Since Eqs. (12) do not contain $\dot{\mathbf{x}}$ and $\dot{q}^4$, they are constraints between the momenta and the coordinates. It therefore follows that Hamilton's equations based on the usual Hamiltonians corresponding to the above non-standard Lagrangians are not the equations of motion. To overcome this difficulty, Dirac's theory of constrained dynamics [23] is applied, which yields the Dirac Hamiltonians

$$H_\nu = e_\nu\phi_\nu^\star + \mathbf{v}_{\mathbf{g}\nu}\cdot[\mathbf{p} - (e_\nu/c)\mathbf{A}_\nu^\star] + V^4 p_4. \tag{13}$$

Particular solutions of the equations of motion following from the Hamiltonians (13) are the constraints (12). The distribution functions $f_\nu(\mathbf{x},q^4,\mathbf{p},p_4,t)$ must guarantee that these constraints are satisfied. As concerns this requirement, it is important to note that $\mathbf{p} - (e_\nu/c)\mathbf{A}_\nu^\star = 0$ and $p_4 = 0$ do not represent special values of some constants of motion. Therefore, δ functions of the constraints are not constants of motion either. But f_ν must be proportional to such δ functions and, at the same time, also a constant of motion. Both conditions are uniquely satisfied by

$$f_\nu = \delta(p_4)\,\delta\!\left(\mathbf{p} - \frac{e_\nu}{c}\mathbf{A}_\nu^\star\right) B_{\nu\parallel}^\star f_{g\nu}(\mathbf{x},q^4,\mu,t), \tag{14}$$

where the guiding center distribution functions $f_{g\nu}$ are constants of motion and solutions of the drift kinetic differential equations

$$\frac{\partial f_{g\nu}}{\partial t} + \mathbf{v}_{g\nu}\cdot\frac{\partial f_{g\nu}}{\partial \mathbf{x}} + V^4\frac{\partial f_{g\nu}}{\partial q^4} = 0. \tag{15}$$

In the present paper cylindrical equilibria are considered. With the coordinates q^1, q^2, and q^3 specified to be the cylindrical coordinates r, θ, and z with unit basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$, the equilibrium vector potential and magnetic field are given by

$$\mathbf{A}^{(0)}=A_\theta^{(0)}(r)\mathbf{e}_\theta+A_z^{(0)}(r)\mathbf{e}_z \tag{16}$$

and

$$\mathbf{B}^{(0)}=B_\theta^{(0)}(r)\mathbf{e}_\theta+B_z^{(0)}(r)\mathbf{e}_z, \tag{17}$$

with

$$\frac{1}{r}(rA_\theta^{(0)})'=B_z^{(0)}, \quad (A_z^{(0)})'=-B_\theta^{(0)}, \tag{18}$$

where the prime denotes differentiation with respect to r. The equilibrium electric field can be expressed in terms of the scalar potential $\phi^{(0)}(r)$ as

$$\mathbf{E}^{(0)}=-\boldsymbol{\nabla}\phi^{(0)}=-(\phi^{(0)})'\mathbf{e}_r. \tag{19}$$

For the equilibria defined above, the guiding center velocity [Eq. (10)] becomes

$$\begin{aligned}\mathbf{v}_{g\nu}^{(0)}&=v_\parallel\mathbf{b}^{(0)}+\mathbf{v}_\perp^{(0)}\\&=v_\parallel\mathbf{b}^{(0)}-\frac{c}{e_\nu B_{\nu\parallel}^{\star(0)}}\left[e_\nu(\phi^{(0)})'+\mu(B^{(0)})'-\frac{e_\nu v_\parallel B_{\nu\perp}^{\star(0)}}{c}\right.\\&\quad\left.+\frac{m_\nu}{2}(v_E^{(0)2})'\right](\mathbf{e}_r\times\mathbf{b}^{(0)}),\end{aligned} \tag{20}$$

where $\mathbf{v}_\perp^{(0)}=\mathbf{b}^{(0)}\times(\mathbf{v}_{g\nu}^{(0)}\times\mathbf{b}^{(0)})$ is the perpendicular component of $\mathbf{v}_{g\nu}^{(0)}$ consisting of the $\mathbf{E}\times\mathbf{B}$, $\boldsymbol{\nabla}B$, curvature, and polarization drifts; $B_{\nu\parallel}^{\star(0)}\equiv\mathbf{B}_\nu^{\star(0)}\cdot\mathbf{b}^{(0)}$ and $B_{\nu\perp}^{\star(0)}\equiv\mathbf{b}^{(0)}\times(\mathbf{B}_\nu^{\star(0)}\times\mathbf{b}^{(0)})$. For thermal particles it holds that $B_{\nu\parallel}^{\star(0)}\approx B^{(0)}$ and $B_{\nu\perp}^{\star(0)}/B_{\nu\parallel}^{\star(0)}\approx O(r_{g\nu}/r_0)$, where $r_{g\nu}$ is the thermal Larmor radius for the particle species ν and r_0 the macroscopic scale length. $\mathbf{v}_{g\nu}^{(0)}$ has no r component, and therefore r is a constant of motion. Since there is also no force parallel to $\mathbf{B}^{(0)}$, another constant of motion is the parallel guiding center velocity $v_\parallel$. The guiding center distribution functions $f_{g\nu}^{(0)}$ are therefore functions of r, $v_\parallel$, and the magnetic moment μ. From Eq. (11) it follows that $V_4^{(0)}=0$, and hence the Dirac Hamiltonians [Eq. (13)] are written in the form

$$H_\nu^{(0)}=e_\nu\phi_\nu^{\star(0)}+\mathbf{v}_{g\nu}^{(0)}\cdot[\mathbf{p}-(e_\nu/c)\mathbf{A}_\nu^{\star(0)}]. \tag{21}$$

The general expression for the second-order perturbation energy [Eq. (3)] is evaluated for these equilibria and for initial perturbations $\mathbf{A}^{(1)}=\mathbf{0}$ and $\dot{\mathbf{A}}^{(1)}=\mathbf{0}$. It is also shown *a posteriori* that one can choose initial perturbations such that the charge density $\rho^{(1)}$ vanishes without changing the particle contributions to the energy. Thus, choosing perturbations of this kind, we can set from the outset $F_{\mu\lambda}^{(1)}=A_\rho^{(1)}=0$.

After a lengthy derivation, which can be conducted along the lines of that for cylindrical equilibria with $\mathbf{E}=\mathbf{0}$ reported in detail in Appendix A of Ref. [19], $F^{(2)}$ is cast in the concise form

$$\begin{aligned}F^{(2)}=-\sum_\nu\int S(r)dr\,dv_\parallel d\mu\Bigg\{&\frac{B_{\nu\parallel}^{\star(0)}}{m_\nu}|G_\nu^{(1)}|^2(\mathbf{k}\cdot\mathbf{v}_{g\nu}^{(0)})\\&\times\left[(\mathbf{b}^{\star(0)}\cdot\mathbf{k})\frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel}-\frac{k_\perp}{\omega_\nu^{\star(0)}}\frac{\partial f_{g\nu}^{(0)}}{\partial r}\right]\Bigg\}.\end{aligned} \tag{22}$$

Here $\omega_\nu^{\star(0)}\equiv(e_\nu B_{\nu\parallel}^{\star(0)})/(cm_\nu)$, $G_\nu^{(1)}(r,q^4,\mu)$ are arbitrary first-order quantities relating to the generating functions $S_\nu^{(1)}$ for the perturbations; $\mathbf{b}_\nu^{\star(0)}\equiv\mathbf{B}_\nu^{\star(0)}/B_{\nu\parallel}^{\star(0)}$; $\mathbf{k}=k_\theta\mathbf{e}_\theta+k_z\mathbf{e}_z$ is the wave vector lying in magnetic surfaces; $k_\parallel$ and $k_\perp$ are its components parallel and perpendicular to $\mathbf{B}^{(0)}$, respectively; and

$$S(r)\equiv r\int_{\theta_0}^{\theta_0+(2\pi/rk_\theta)}\int_{z_0}^{z_0+(2\pi/k_z)}d\theta\,dz$$

is a normalization surface, where θ_0 and z_0 are constants. We note that $F^{(2)}$ depends on $G_\nu^{(1)}$ only via $|G_\nu^{(1)}|^2$. The first-order charge density $\rho^{(1)}$ is a $v_\parallel$, μ integral over an expression that is linear in $G_\nu^{(1)}$. One can therefore satisfy the relation $\rho^{(1)}=0$ by a proper distribution of positive and negative values of $G_\nu^{(1)}$ on which $F^{(2)}$ does not depend.

Compared with the corresponding expression for equilibria with $\mathbf{E}^{(0)}=\mathbf{0}$ [Eq. (37) of Ref. [18]], $F^{(2)}$ contains terms stemming from $\mathbf{b}_\nu^{\star(0)}$ and from the $\mathbf{E}\times\mathbf{B}$ and polarization drift components of $\mathbf{v}_{g\nu}$. In particular, as shown in Sec. III the $\mathbf{E}\times\mathbf{B}$ drift modifies the condition for the existence of NEP's with wave vectors perpendicular to $\mathbf{B}^{(0)}$.

III. CONDITIONS FOR THE EXISTENCE OF NEGATIVE-ENERGY PERTURBATIONS

The derivations in this section are very similar to those concerning equilibria with $\mathbf{E}=\mathbf{0}$ [17–19] so that details need not be given here. The following conditions must be satisfied only locally in r, $v_\parallel$ and μ, and refer to the frame of reference of minimum energy.

Parallel perturbations ($k_\perp=0$): NEP's exist when

$$v_\parallel\frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel}>0 \tag{23}$$

is satisfied for at least one particle species. This condition, which was first derived by Pfirsch and Morrison for a homogeneous magnetized plasma [7], agrees with those obtained by Correa-Restrepo and Pfirsch for several Maxwell-Vlasov equilibria [8–12].

Oblique perturbations ($k_\parallel\neq0$ and $k_\perp\neq0$): If condition (23) is satisfied for at least one particle species ν, only perturbations with wave vectors satisfying in addition the relations

$$\frac{k_\parallel}{k_\perp}<\min(\Lambda_\nu,M_\nu) \quad\text{or}\quad \frac{k_\parallel}{k_\perp}>\max(\Lambda_\nu,M_\nu), \tag{24}$$

with

$$\Lambda_\nu\equiv-\frac{\mathbf{v}_{g\nu\perp}^{(0)}}{v_\parallel}\cdot(\mathbf{e}_r\times\mathbf{b}^{(0)})$$

and

$$M_\nu \equiv \left(\frac{1}{\omega_\nu^{\star(0)}} \frac{\partial f_{g\nu}^{(0)}}{\partial r} - \frac{B_{\nu\perp}^{\star(0)}}{B_{\nu\parallel}^{\star(0)}} \frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel} \right) \left(\frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel} \right)^{-1},$$

can have negative energy. The orders of magnitude of Λ_ν and M_ν depend on the particle energy. For example, if

$$m_\nu v_\parallel^2 \approx \mu B^{(0)} \approx |e_\nu \phi| \approx T_\nu, \tag{25}$$

with $\phi(\infty)=0$, it holds that

$$|\Lambda_\nu| \approx |M_\nu| \approx \frac{r_{g\nu}}{r_0} \ll 1. \tag{26}$$

Relation (26) indicates that condition (24) imposes no essential restriction on the magnitude or the orientation of the wave vectors associated with NEP's.

If

$$v_\parallel \frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel} < 0, \tag{27}$$

a condition which is satisfied at all points of a Maxwellian distribution function, NEP's exist if, in addition to Eq. (27), it holds that

$$\min(\Lambda_\nu, M_\nu) < \frac{k_\parallel}{k_\perp} < \max(\Lambda_\nu, M_\nu). \tag{28}$$

For the scaling (25), condition (28) implies that

$$\frac{k_\parallel}{k_\perp} \approx \frac{r_{g\nu}}{r_0} \ll 1, \tag{29}$$

which indicates that the most important NEP's, in the sense that the less restrictive condition (27) is involved, are associated with nearly perpendicular wave vectors. It may be noted that for a homogeneous magnetized plasma in thermal equilibrium, although condition (27) is satisfied, NEP's are not possible because $M_\nu = \Lambda_\nu = 0$ and therefore condition (28) is not satisfied. This also follows from condition (23) which is pertinent for the existence of NEP's in a homogeneous magnetized plasma and is not satisfied in thermal equilibrium, i.e., for Maxwellian distribution functions.

Perpendicular perturbations ($k_\parallel = 0$): In this case the second-order perturbation energy [Eq. (22)] reduces to

$$F^{(2)} = 4\pi \sum_\nu \int dr\, dv_\parallel d\mu\, S(r) |G_\nu^{(1)}|^2 \times \frac{B_{\nu\parallel}^{\star(0)}}{m_\nu^2} \frac{W_{\nu\perp}}{(B^{(0)})^2} \left(\frac{k_\perp}{\omega_\nu^{\star(0)}} \right)^2 Z_\nu Q_\nu, \tag{30}$$

with

$$Z_\nu = \frac{(B^{(0)})^2}{4\pi W_{\nu\perp}} \left[\frac{e_\nu}{c} v_\parallel B_{\nu\perp}^{\star(0)} - e_\nu \phi' - \mu (B^{(0)})' - \frac{m_\nu}{2} (v_E^{(0)2})' \right] \tag{31}$$

and

$$Q_\nu = \frac{\partial f_{g\nu}^{(0)}}{\partial r} - \omega_\nu^{\star(0)} \frac{B_{\nu\perp}^{\star(0)}}{B_{\nu\perp}^{\star(0)}} \frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel}, \tag{32}$$

where $W_{\nu\perp} = \mu B^{(0)}$ is the perpendicular particle energy. Equation (30) implies that $F^{(2)} < 0$ for *any* $k_\perp$ whenever the condition

$$Z_\nu Q_\nu < 0 \tag{33}$$

is satisfied *irrespective of the sign of* $v_\parallel (\partial f_{g\nu}^{(0)} / \partial v_\parallel)$. Therefore, there are two regimes of NEP's which are determined by the relations

$$Z_\nu < 0$$

and

$$Q_\nu > 0 \tag{34}$$

and

$$Z_\nu > 0$$

and

$$Q_\nu < 0. \tag{35}$$

For the evaluation of conditions (34) and (35) the equilibrium equations are required, which are constructed in Sec. IV.

IV. QUASINEUTRAL MAXWELL-DRIFT KINETIC EQUILIBRIUM EQUATIONS

The equilibria must satisfy

$$\nabla \cdot \mathbf{E}^{(0)} = 4\pi \rho^{(0)} \tag{36}$$

and

$$\nabla \times \mathbf{B}^{(0)} = \frac{4\pi}{c} \mathbf{j}^{(0)}, \tag{37}$$

where the charge density $\rho^{(0)}$ and current density $\mathbf{j}^{(0)}$ are expressed self-consistently in terms of the guiding center distributions functions $f_{g\nu}^{(0)}$ in the context of the Maxwell-drift kinetic theory [see Eqs. (8.14) and (8.15) of Ref. [24]]. For the system under consideration, owing to the presence of $\mathbf{E}^{(0)}$, the set of equilibrium equations following from Eqs. (36) and (37) are rather complicated. For this reason we employ the quasineutral Maxwell-drift kinetic theory which can be derived self-consistently by dropping the electric-field-energy term in the Lagrangian (6). A similar method was employed in Refs. [14,20,25]. Consequently, Eq. (36) is replaced by the quasineutrality condition, which is explicitly given by

$$\sum_\nu e_\nu \int dv_\parallel d\mu B_{\nu\parallel}^{\star(0)} f_{g\nu}^{(0)} + \sum_\nu div \int dv_\parallel d\mu B_{\nu\parallel}^{\star(0)} f_{g\nu}^{(0)} \frac{m_\nu c}{B^{(0)}} (\mathbf{v}_{g\nu}^{(0)} - \mathbf{v}_E^{(0)}) \times \mathbf{b}^{(0)} = 0, \tag{38}$$

and Eq. (37) by

$$\sum_\nu e_\nu \int dv_\parallel d\mu B^{\star(0)}_{\nu\parallel} f^{(0)}_{g\nu} \mathbf{v}^{(0)}_{g\nu} - \sum_\nu c \ \text{curl} \ \int dv_\parallel d\mu B^{\star(0)}_{\nu\parallel} f^{(0)}_{g\nu} \Bigg[\mu \mathbf{b}^{(0)} - \frac{m_\nu}{B^{(0)}} v_\parallel (\mathbf{v}^{(0)}_{g\nu\perp} - \mathbf{v}^{(0)}_E) - \frac{m_\nu c}{(B^{(0)})^2} (\mathbf{v}^{(0)}_{g\nu} - \mathbf{v}^{(0)}_E) \times \mathbf{E}^{(0)} + \frac{2m_\nu}{B^{(0)}} \{(\mathbf{v}^{(0)}_{g\nu} - \mathbf{v}^{(0)}_E) \cdot \mathbf{v}^{(0)}_E\} \mathbf{b}^{(0)} \Bigg] = \frac{c}{4\pi} \nabla \times \mathbf{B}^{(0)}. \tag{39}$$

The first terms on the left-hand sides of Eqs. (38) and (39) represent guiding center charge and current density contributions, respectively, and the other terms polarization and magnetization contributions. We consider equilibria of the following kinds.

(1) The distribution functions are specified to be local shifted Maxwellians,

$$f^{(0)}_{g\nu} = \left(\frac{m_\nu}{2\pi} \right)^{1/2} \frac{N^{(0)}_\nu(r)}{T^{(0)}_\nu(r)^{3/2}} \times \exp\left\{ - \frac{\mu B^{(0)}(r) + 1/2 m_\nu [v_\parallel - V^{(0)}_\nu(r)]^2}{T^{(0)}_\nu(r)} \right\}, \tag{40}$$

where $N^{(0)}_\nu$ and $T^{(0)}_\nu$ are, respectively, the density and temperature for particles of species ν. They can describe cylindrical tokamaklike, reversed-field-pinchlike, and, for $V^{(0)}_\nu \equiv 0$, shearless stellaratorlike plasmas, which are close to thermal equilibrium. For the former equilibria the shift velocities $V^{(0)}_\nu$ satisfy

$$\frac{V^{(0)}_\nu}{(v_\nu)_{th}} \approx \frac{r_{g\nu}}{r_0} \ll 1 \tag{41}$$

and, as shown later, lead to a nonvanishing ‘‘toroidal current.’’

(2) Since a radial electric field may play a role in the L-H transition of magnetic confinement systems, e.g., Refs. [26], [27], for the ion electrostatic energy we adopt the scaling

$$|e_i \phi^{(0)}| \approx T^{(0)}_i, \quad \phi^{(0)}(\infty) = 0, \tag{42}$$

which is satisfied in the edge region.

Using the above assumptions, neglecting small terms of the order $r_{g\nu}/r_0$ and suppressing the superscript (0) from the equilibrium quantities, Eq. (38) and the θ and z components of Eq. (39), respectively, yield

$$\sum_\nu e_\nu N_\nu = 0, \tag{43}$$

$$j_\theta = b_\theta e_i N_i (V_i - V_e) + c \frac{b_z}{B} P' = - \frac{c}{4\pi} B'_z, \tag{44}$$

and

$$j_z = b_z e_i N_i (V_i - V_e) - c \frac{b_\theta}{B} P' = - \frac{c}{4\pi} \frac{1}{r} (r B_\theta)', \tag{45}$$

where

$$P \equiv \sum \int dv_\parallel d\mu B^\star_{\nu\parallel} \mu B f_{g\nu} = \sum_\nu N_\nu T_\nu. \tag{46}$$

For $V_\nu \equiv 0$ for all ν, Eqs. (44) and (45), respectively, reduce to

$$\frac{b_z}{B} P' = - \frac{B'_z}{4\pi} \tag{47}$$

and

$$\frac{b_\theta}{B} P' = - \frac{1}{4\pi} \frac{1}{r} (r B_\theta)'. \tag{48}$$

The solutions of Eqs. (47) and (48) satisfy the relation $B_\theta = a(B_z/r)$, with $a = \text{const}$. They are singular at $r=0$ and therefore $a=0$. For $B_\theta \equiv 0$, Eq. (48) is satisfied identically, and the only possible equilibrium, which is described by Eq. (47), is a θ-pinch or shearless stellaratorlike configuration with vanishing axial current, a case which is examined in Sec. VI. Multiplying Eqs. (44) and (45) by the integrating factors B_z and B_θ, respectively, and adding the resulting equations, one obtains the pressure balance relation

$$\frac{d}{dr} \left(P + \frac{B^2}{8\pi} \right) + \frac{B^2_\theta}{4\pi r} = 0, \tag{49}$$

which will be used in place of Eq. (45).

Summarizing, quasineutral equilibria can be described by the set of Eqs. (43), (44), (46), and (49). Four out of the eight functions involved must be assigned, e.g., $P(r)$, $B_z(r)$, the shift velocity difference $V_i(r) - V_e(r)$ and $T_i(r)$; then $B_\theta(r)$ can be obtained from Eq. (49), $N_i(r)$ from Eq. (44), $N_e(r)$ from Eq. (43), and $T_e(r)$ from Eq. (46). Analytic solutions, which are required for determining the portion of active particles in equilibria of magnetic confinement systems, are constructed in Sec. VI.

V. PERPENDICULAR NEP's IN EQUILIBRIA OF MAGNETIC CONFINEMENT SYSTEMS

In this section condition (33) for the existence of perpendicular NEP's is applied to the equilibria defined in Secs. II and IV. For distribution functions of the form (40) and the pressure balance relation (49), the quantity Q_ν [Eq. (32)] reduces to

$$Q_\nu = \frac{N'_\nu}{N_\nu} U_\nu, \tag{50}$$

where

$$U_\nu \equiv 1 - \tfrac{2}{3} + \eta_\nu \epsilon_{1\nu} + \epsilon_{2\nu}, \tag{51}$$

$$\epsilon_{1\nu} \equiv \frac{W_{\nu\perp}}{T_\nu} \left(1 + \frac{W_{\nu\parallel}}{W_{\nu\perp}} \right), \tag{52}$$

$$\epsilon_{2\nu} \equiv \frac{4\pi}{B^2}\frac{W_{\nu\perp}}{T_\nu}\frac{N_\nu}{N'_\nu}R_\nu, \tag{53}$$

and

$$R_\nu \equiv P' + \frac{B_\theta^2}{4\pi r}\left(1+2\frac{W_{\nu\parallel}}{W_{\nu\perp}}\right). \tag{54}$$

Depending on the value of T_i/T_e, the effect of **E** on perpendicular NEP's is examined in the following two regions.

A. $T_i/T_e < \beta_c \approx P/(B^2/8\pi)$

Assuming the scaling (42) to hold it can be shown that the pressure gradient and a term relating to the curvature of B_θ dominate in Z_ν, i.e.,

$$Z_\nu \approx P' + \frac{B_\theta^2}{4\pi r}\left(1+2\frac{W_{\nu\parallel}}{W_{\nu\perp}}\right) = R_\nu. \tag{55}$$

Condition (33) can be put into the form

$$R_\nu \frac{N'_\nu}{N_\nu} U_\nu < 0. \tag{56}$$

Relation (56) is identical to the corresponding one in equilibria with $\mathbf{E}=\mathbf{0}$ [relation (58) of Ref. [18]]. For singly peaked density and the temperature profiles, and therefore $\eta_\nu > 0$ for all ν, which is the most common case in equilibria of magnetic confinement systems, there are two regimes of NEPs depending on the sign of R_ν [Eq. (54)].

1. $R_\nu < 0$

Condition (56) implies that $U_\nu < 0$ must hold. The last two terms of U_ν [Eqs. (51)–(53)] become non-negative and vanish for $W_{\nu\parallel} \rightarrow 0$ and $W_{\nu\perp} \rightarrow 0$. Consequently, $U_\nu < 0$ is satisfied whenever

$$\eta_\nu > \tfrac{2}{3}. \tag{57}$$

The existence of perpendicular ion NEP's for any $k_\perp$ is therefore related to the threshold value of $\frac{2}{3}$ of the quantity η_ν. As discussed in Ref. [17], this threshold value is subcritical in the sense that it is lower than the critical value $\eta_\nu^c \approx 1$ for linear stability of temperature-gradient-driven modes.

2. $R_\nu > 0$

The condition for the existence of perpendicular NEP's becomes $U_\nu > 0$. In this case the quantity $\eta_\nu \epsilon_{1\nu} + \epsilon_{2\nu}$ can be either positive or negative, and therefore no restriction is imposed on η_ν. It may be noted that for plane equilibria it holds that $R_\nu = P' < 0$ and therefore the second regime of NEP's is associated with the curvature of B_θ.

B. $T_i/T_e > \beta_c$

If the scaling (42) holds, the term $e_\nu \phi'$ related to the $\mathbf{E}\times\mathbf{B}$ drift dominates in Z_ν [Eq. (31)], i.e.,

$$Z_\nu \approx -\frac{B^2}{4\pi W_{\nu\perp}} e_\nu \phi'. \tag{58}$$

Condition (33) then becomes

$$e_\nu \phi' \frac{N'_\nu}{N_\nu} U_\nu > 0. \tag{59}$$

Relation (59) shows that the existence of perpendicular NEP's depends on the sign of the particle species charge and the polarity of **E**. Henceforth and up to Sec. VI C, $\mathbf{E}\neq\mathbf{0}$ will refer to this case $(T_i/T_e > \beta_c)$. In the edge region the radial electric field is usually negative [26,27]. It is noted here that the impact of the polarity of an externally induced radial electric field was investigated experimentally [28]. It was found that whereas the energy confinement in H modes with $\mathbf{E}>\mathbf{0}$ is at least as good as in those with $\mathbf{E}<\mathbf{0}$, the ratio of the ion confinement time to the energy confinement time is about three times lower in the former case. We examine therefore in the following NEP's for ions and electrons in equilibria with $\phi' > 0$.

1. Ions

In this case $e_\nu \phi'$ is positive, and condition (59) is satisfied whenever $U_i < 0$. Depending on the sign of R_i, there are two regimes of NEP's: (a) If $R_i < 0$, $U_i < 0$ is satisfied whenever $\eta_i > \frac{2}{3}$, and (b) if $R_i > 0$, no restriction is imposed on η_i. It is pointed out, however, that for $\mathbf{E}=\mathbf{0}$ the condition associated with this second regime is $U_i > 0$. As shown in Sec. VI, this difference affects the fraction of active ions.

2. Electrons

Condition (59) is satisfied whenever $U_e > 0$. This yields

$$\eta_e < \tfrac{2}{3}(1+\eta_e \epsilon_{1e} + \epsilon_{2e}). \tag{60}$$

For cold electrons ($W_{e\parallel} \rightarrow 0$ and $W_{e\perp} \rightarrow 0$), condition (60) implies that NEP's exist whenever $\eta_e < \frac{2}{3}$. This indicates that **E** has a ''stabilizing'' effect on electron NEP's for large values of η_e. Owing to hot electrons, however, condition (60) does not yield an upper threshold value of η_e because electrons with nonvanishing energy activate NEP's in the regime where $\eta_e > \frac{2}{3}$. To determine the value of η_e for which half of the electrons are active, condition (60) is written in the form

$$C_e(r)\frac{W_{e\parallel}}{T_e} + D_e(r)\frac{W_{e\perp}}{T_e} > \frac{3}{2} - \frac{1}{\eta_e}, \tag{61}$$

where

$$C_e(r) \equiv \left(1 + \frac{2}{r}\frac{B_\theta^2}{B^2}\frac{T_e}{T'_e}\right)$$

and

$$D_e(r) \equiv \left[1 + \frac{4\pi}{B^2}\frac{T_e}{T'_e}\left(P' + \frac{B_\theta^2}{4\pi r}\right)\right]. \tag{62}$$

For a magnetic confinement system, it holds that $B_\theta^2/B^2 \approx P/(B^2/8\pi) \equiv \beta$, with max $\beta \approx 0.1$, and therefore $C_e \approx D_e \approx 1$. Consequently, condition (61) implies that nearly half of the electrons are active whenever it holds that $3/2 - 1/\eta_e \approx \frac{3}{4}$. This yields

$$\eta_e^0 \approx \tfrac{4}{3}. \tag{63}$$

Therefore, if $\mathbf{E} \neq \mathbf{0}$, less than half of the electrons are active whenever $\eta_e > \eta_e^0$, and this portion decreases as η_e takes larger values. On the other side, if $\mathbf{E}=\mathbf{0}$, more than half of the electrons are active whenever $\eta_e > \eta_e^0$, with this portion increasing as η_e takes larger values.

VI. ANALYTIC EQUILIBRIUM SOLUTIONS AND ACTIVE PARTICLES

In this section the portion of active particles is determined on the basis of analytic shearless stellaratorlike, tokamaklike, and reversed-field-pinchlike equilibrium solutions.

A. Shearless stellaratorlike (θ pinch) equilibria

We consider the following profiles:

$$P = P(0)(1-\rho^2), \tag{64}$$

$N_i = N_i(0)(1-\rho^2)^\xi$ and $T_i = T_i(0)(1-\rho^2)^{1-\xi}$, where $\rho \equiv r/r_0$, and r_0 corresponds to the plasma surface. Equations (49) with $B_\theta \equiv 0$, $\Sigma_\nu e_\nu N_\nu = 0$, and $P = \Sigma_\nu N_\nu T_\nu$, then yield

$$B_z = B_{z0}[1-\beta_0(1-\rho^2)]^{1/2},$$
$$\mathbf{j} = \boldsymbol{\nabla} B_z \times \mathbf{e}_z = -\frac{B_{z0}}{r_0}\,\beta_0\,\frac{\rho}{[1-\beta_0(1-\rho^2)]^{1/2}}\,\mathbf{e}_\theta, \tag{65}$$

$N_e = N_e(0)(1-\rho^2)^\xi$ and $T_e = T_e(0)(1-\rho^2)^{1-\xi}$. Here B_{z0} is the external constant ''toroidal'' magnetic field, $\beta_0 \equiv P(0)/(B^2/8\pi)$, and the parameter ξ $(0 \leqslant \xi \leqslant 1)$ determines equilibria with different values of η_ν, i.e.,

$$\eta_\nu \equiv \frac{\partial \ln T_\nu}{\partial \ln N_\nu} = \frac{1-\xi}{\xi}. \tag{66}$$

Ion and electron NEP's are now examined separately.

1. Ions

Since the magnetic field lines are straight, it holds that $R_i = P' < 0$, and therefore that ion NEP's exist only in equilibria with $\eta_i > \frac{2}{3}$. The pertinent condition $U_i < 0$ becomes

$$\frac{W_{i\parallel}}{T_i} + \left[1 + \frac{\beta}{2(1-\xi)}\,(1-\rho^2)\right]\frac{W_{i\perp}}{T_i} < \frac{3}{2} - \frac{1}{\eta_i}. \tag{67}$$

Relation (67) implies the following.

(1) The portion of active ions increases as η_i takes larger values. In particular, for a flat temperature and peaked density profile there are no active ions; for $\eta_i = 1$, one-third of the thermal ions are active, and for $\eta_i = 2$ this fraction becomes $\frac{2}{3}$; for a flat density and a peaked temperature profile $(\eta_i \to \infty)$, all ions are active;

(2) The portion of active ions increases from the center $\rho=0$ to the edge region $\rho=1$.

2. Electrons

For $\mathbf{E}=\mathbf{0}$, the situation is similar to the foregoing one for ions. For $\mathbf{E} \neq \mathbf{0}$ the condition for the existence of electron NEP's is $U_e > 0$, and therefore the fractions of active electrons and ions are complementary to each other. Thus, as also discussed in Sec. V, the electric field stabilizes electron NEP's for $\eta_e > \frac{4}{3}$, e.g., for the equilibrium profiles (64)–(66), one-third of the thermal electrons are active when $\eta_e = 2$, while the corresponding fraction for the equilibrium with $\mathbf{E}=\mathbf{0}$ is $\frac{2}{3}$. In addition, the fraction of active electrons *decreases* from the center to the edge. This indicates that in the presence of $\mathbf{E}$ self-sustained turbulence associated with electron NEP's should be reduced in the edge.

B. Tokamaklike (screw pinch) equilibria

The following profiles correspond to a special solution of Eq. (49):

$$B_z = [B_z^2(0) + 8\pi P(0)(1-\alpha^2)\rho^2]^{1/2}, \tag{68}$$

where α is a parameter which describes the magnetic properties of the plasma, i.e., the plasma is diamagnetic for $\alpha < 1$ and paramagnetic for $\alpha > 1$;

$$B_\theta = 2\sqrt{\pi P(0)}\,\alpha\rho \tag{69}$$

is the constant axial current density; $N_\nu = N_\nu(0)(1-\rho^2)^\xi$; and $T_\nu = T_\nu(0)(1-\rho^2)^{1-\xi}$, with $\nu = i,e$. Ion and electron NEP's are now examined for $\eta_\nu = 1$, which is close to linear stability threshold for gradient temperature driven modes.

1. Ions

For $\mathbf{E}=\mathbf{0}$ the portion of active ions is determined by conditions (34) and (35) which, respectively, become

$$\frac{W_{i\parallel}}{W_{i\perp}} < \frac{1}{2}\left(\frac{2}{\alpha^2} - 1\right)$$

and

$$[1 - \tfrac{1}{2}\beta(1-\rho^2)(\alpha^2-2)]\,\frac{W_{i\perp}}{T_i} + [1 - \beta\alpha^2(1-\rho^2)]\,\frac{W_{i\parallel}}{T_i} < \frac{1}{2} \tag{70}$$

and

$$\frac{W_{i\parallel}}{W_{i\perp}} > \frac{1}{2}\left(\frac{2}{\alpha^2} - 1\right)$$

and

$$[1 - \tfrac{1}{2}\beta(1-\rho^2)(\alpha^2-2)]\,\frac{W_{i\perp}}{T_i} + [1 - \beta\alpha^2(1-\rho^2)]\,\frac{W_{i\parallel}}{T_i} > \frac{1}{2}. \tag{71}$$

Relations (70) and (71) imply the following.

(1) The portion of active ions increases with α, i.e., it is smaller in a diamagnetic system and larger in a paramagnetic

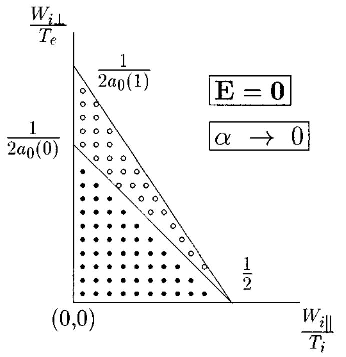

FIG. 1. The portion of active ions for a strongly diamagnetic plasma with $\mathbf{E}=\mathbf{0}$ and $\eta_i=1$ which is deduced from Eq. (70) $[\alpha_0(\rho)\equiv 1+\beta(1-\rho^2)]$. The dotted area stands for the active particles at the center ($\rho=0$), while the area filled by circles for the additional active particles at the edge ($\rho=1$).

system. The particular cases of a strongly diamagnetic plasma ($\alpha\to 0$), of an equilibrium with constant ''toroidal'' magnetic field ($\alpha^2=1$), and of a paramagnetic plasma ($\alpha^2=2$) are illustrated in Figs. 1, 2, and 3, respectively. The fractions of active ions are nearly $\frac{1}{3}$ for $\alpha\to 0$, $\frac{1}{2}$ for $\alpha^2=1$, and $\frac{2}{3}$ for $\alpha^2=2$. It is noted that for $\alpha\to 0$ only the branch (70), associated with the threshold value $\eta_i=\frac{2}{3}$, contributes, while for $\alpha^2=2$ exclusively the branch (71) associated with the curvature of the poloidal field lines contributes.

(2) In all regimes the fraction of active ions increases from the center to the edge. In Figs. 1, 2, and 3 the dotted area stands for the active particles at the center ($\rho=0$), while the area filled by circles for the additional active particles at the edge ($\rho=1$).

It is noted here that for $\mathbf{E}=\mathbf{0}$ similar results hold for electrons. For $\mathbf{E}\neq\mathbf{0}$, active ions obtain from condition $U_i<0$ (irrespective of the sign of R_i), which leads to

$$\left[1-\frac{1}{2}\,\beta(1-\rho^2)(\alpha^2-2)\right]\frac{W_{i\perp}}{T_i}+[1-\beta\alpha^2(1-\rho^2)]\,\frac{W_{i\parallel}}{T_i}<\frac{1}{2}. \qquad (72)$$

Relation (72) implies the following.

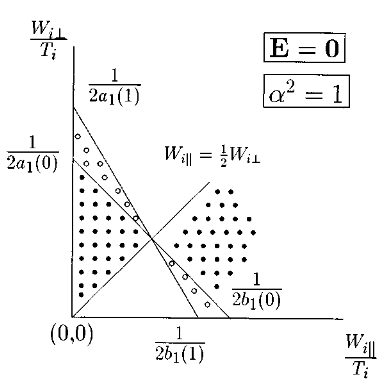

FIG. 2. The portion of active ions for the equilibrium with $\mathbf{E}=\mathbf{0}$, $\eta_i=1$, and $B_z=$const, which is deduced from Eqs. (70) and (71) $[\alpha_1(\rho)\equiv 1+(\beta/2)(1-\rho^2),\ b_1(\rho)\equiv 1-\beta(1-\rho^2)]$.

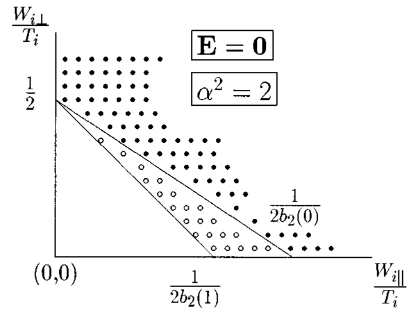

FIG. 3. The portion of active ions for the equilibrium of a paramagnetic plasma with $\mathbf{E}=\mathbf{0}$ and $\eta_e=1$, which is deduced from Eq. (71) $[b_2(\rho)\equiv 1-2\beta(1-\rho^2)]$.

(1) The portion of active ions is nearly independent of the magnetic properties of the plasma; it is approximately $\frac{1}{3}$ for any value of α.

(2) The portion of active ions can either be nearly independent of ρ, e.g., for an equilibrium with constant B_z ($\alpha^2=1$) (Fig. 4) or decreases from the center to the edge, e.g., for a paramagnetic plasma $\alpha^2=2$ (Fig. 5), while this portion always increases for equilibria with $\mathbf{E}=\mathbf{0}$.

Thus $\mathbf{E}$ leads to a reduction of active ions.

2. Electrons

Recalling that the portion of active electrons is the same as that of active ions when $\mathbf{E}=\mathbf{0}$, and complementary when $\mathbf{E}\neq\mathbf{0}$, respectively, the former portion can be determined on the basis of the foregoing analysis for ions. Thus, in addition to the stabilizing effect of $\mathbf{E}$ for $\eta_e>\frac{4}{3}$, the fraction of active electrons (a) becomes nearly independent of the magnetic properties of the plasma, and (b) can decrease from the center to the edge, e.g., for the most common case of a diamagnetic plasma.

C. Reversed-field-pinchlike (force-free) equilibria

The solution of Eq. (49) with $P'=0$ leads to $B_z=B_z(0)J_0(\rho)$ and $B_\theta=B_z(0)J_1(\rho)$, where J_0 and J_1 are Bessel functions. These profiles satisfactorily describe the central region of the relaxed state of a reversed-field pinch

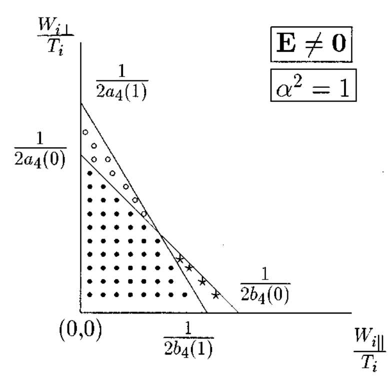

FIG. 4. The portion of active ions for the equilibrium with $\mathbf{E}\neq\mathbf{0}$, $\eta_i=1$ and $B_z=$const, which is deduced from Eq. (72) $[\alpha_4(\rho)\equiv 1+(\beta/2)(1-\rho^2),\ b_4(\rho)\equiv 1-\beta(1-\rho^2)]$. The excess portion at the edge indicated by circles nearly compensates for the excess portion at the center indicated by stars.

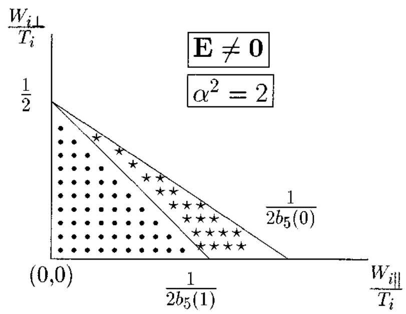

FIG. 5. The portion of active ions for a strongly diamagnetic plasma with $\mathbf{E}\neq\mathbf{0}$ deduced from Eq. (72) $[\alpha_5(\rho)=1-2\beta(1-\rho^2)]$. The area filled by stars represents the excess portion at the plasma center.

[29]. By appropriately assigning $V_i(r)-V_e(r)$, one can derive equilibria with a variety of density and temperature profiles for which NEP's exist and a considerable fraction of active ions and electrons are involved. From the equilibria considered it turns out that **E** (a) does not affect the electron NEP's, and (b) enhances the fraction of active ions.

As an example, we consider the most common equilibrium with constant density and temperature profiles:

$$N_\nu = N_{\nu 0}, \quad T_\nu = T_{\nu 0}. \tag{73}$$

For $\mathbf{E}=\mathbf{0}$, with the aid of relation (55), condition (33) becomes

$$\frac{W_{\nu\perp}}{T_{\nu 0}}\frac{B_\theta^2}{\rho B^2}\left(1+2\frac{W_{\nu\|}}{W_{\nu\perp}}\right)<0 \tag{74}$$

for any particle species ν. Therefore there are neither ion nor electron NEP's.

If $\mathbf{E}\neq\mathbf{0}$, NEP's exist whenever the condition

$$\frac{e_\nu\phi'}{T_{\nu 0}}\frac{B_\theta^2}{\rho B^2}\left(1+2\frac{W_{\nu\|}}{W_{\nu\perp}}\right)>0, \tag{75}$$

following from relations (33) and (58), is satisfied. Owing to the presence of the particle species charge in condition (75), for $\phi'>0$ all ions are active, while the active electrons are not affected.

VII. CONCLUSIONS

The impact of a radial electric field on negative-energy perturbations (NEP's) in cylindrical equilibria of magnetically confined plasmas was investigated within the framework of linearized dissipationless Maxwell-drift kinetic theory. The investigation consisted in evaluating the general expression for the second-order perturbation energy derived by Pfirsch and Morrison for the equilibria under consideration and for vanishing initial-field perturbations; then the conditions for the existence of NEP's were obtained.

The electric field **E** does not affect the following condition for perturbations with wave vectors parallel and oblique to the equilibrium magnetic field ($k_\|\neq 0$): If the equilibrium guiding center distribution function $f_{g\nu}^{(0)}(r,v_\|,\mu)$ of any species ν satisfies the relation $v_\|(\partial f_{g\nu}^{(0)}/\partial v_\|)>0$ locally in r, $v_\|$ and μ, parallel and oblique NEP's exist with no essential restriction on **k**. The condition for the existence of perpendicular NEP's ($k_\|=0$), which holds regardless of the sign of $v_\|(\partial f_{g\nu}^{(0)}/\partial v_\|)$, is modified. For $|e_i\phi|\approx T_i$ the effect of **E** on perpendicular NEP's depends on the value of T_i/T_e, i.e., (a) for $T_i/T_e<\beta_c\approx P/(B^2/8\pi)$, the electric field has no effect; and (b) for $T_i/T_e>\beta_c$, a case which is of operational interest in magnetic confinement systems, the existence of perpendicular NEP's depends on the sign of the particle species charge and the polarity of **E** [relation (59)]. For $\mathbf{E}<\mathbf{0}$, we found the following.

(1) For cylindrical tokamaklike equilibria described by local shifted Maxwellian distribution functions and singly peaked pressure profiles, there exist two regimes of NEP's for both ions and electrons. One regime is associated with the curvature of the poloidal magnetic field. In the other regime the threshold value 2/3 of $\eta_i\equiv\partial\ln T_i/\partial\ln N_i$ is involved for ion NEP's, as in equilibria with $\mathbf{E}=\mathbf{0}$, while a critical value of η_e does not occur for the existence of electron NEP's. However, **E** has the following ''stabilizing'' effects on both particle species:

(a) The portion of particles associated with NEP's (active particles) is nearly independent of the plasma magnetic properties, i.e., it is nearly the same in a diamagnetic and in a paramagnetic plasma, while in equilibria with $\mathbf{E}=\mathbf{0}$ this portion is much larger in a paramagnetic than in a diamagnetic plasma.

(b) The portion of active particles can be either constant or decreases from the center to the edge, e.g., in the case of active electrons of a diamagnetic plasma, while it always increases in the corresponding equilibria with $\mathbf{E}=\mathbf{0}$.

In particular, the fraction of active electrons decreases with increasing η_e and for $\eta_e>\eta_0\approx\frac{4}{3}$ the electric field stabilizes electron NEP's in the sense that the fraction of active electrons becomes smaller than the one corresponding to equilibria with $\mathbf{E}=\mathbf{0}$.

(2) In shearless stellaratorlike equilibria described by local Maxwellian distribution functions and pressure profiles identical to those of tokamaklike equilibria, **E** leads to similar stabilizing effects on electron NEP's, that is, it reduces the fraction of active electrons (a) for $\eta_e>\eta_0\approx\frac{4}{3}$ and (b) from the center to the edge.

In addition, irrespective of the value of T_i/T_e, **E** does not affect electron NEP's in reversed-field-pinchlike equilibria, but ''destabilizes'' the ion NEP's in the sense that it enhances the portion of active ions. For example, for an equilibrium with constant density and temperature profiles all ions are active in the presence of **E**, while there are not active ions when $\mathbf{E}=\mathbf{0}$.

The present results indicate that a radial electric field leads to a reduction of NEP activity in the edge region of tokamaks and stellarators. For electrons, which may mainly contribute to anomalous transport, this reduction is more pronounced.

Finally, it may be noted that according to the results of our previous work [18,19] and in the present study, the curvature of the poloidal magnetic field is unfavorable in the sense that it gives rise to an increase of NEP activity. It can be speculated that this is true for an arbitrary magnetic field configuration. To check this conjecture, it is interesting to investigate NEP's in a toroidal equilibrium, e.g., a tokamak, in which the toroidal magnetic field is favorably curved on the inside and unfavorably on the outside of the torus. Such a study might also reveal the effect of toroidicity on other aspects of NEP's, e.g., the threshold value $\eta_\nu = \frac{2}{3}$.

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[1] A. J. Brizard, J. J. Morehead, and A. N. Kaufman, Phys. Rev. Lett. **77**, 1500 (1996).
[2] J. Weiland and H. Wilhelmson, *Coherent Non-linear Interaction of Waves in Plasmas* (Pergamon, New York, 1977).
[3] H. Wilhelmson, Nucl. Phys. A **518**, 84 (1990).
[4] P. J. Morrison and M. Kotschenreuther, in *Nonlinear World, IV International Workshop on Nonlinear and Turbulent Processes in Physics, Kiev*, edited by V. G. Bar'yakhtar, V. M. Chernousenko, N. S. Erokhin, A. G. Sitenko, and V. E. Zakharov (World Scientific, Singapore, 1990), p. 910.
[5] P. J. Morrison and D. Pfirsch, Phys. Rev. A **40**, 3898 (1989).
[6] P. J. Morrison and D. Pfirsch, Phys. Fluids B **2**, 1105 (1990).
[7] D. Pfirsch and P. J. Morrison, Phys. Fluids B **3**, 271 (1991).
[8] D. Correa-Restrepo and D. Pfirsch, Phys. Rev. A **45**, 2512 (1992).
[9] D. Correa-Restrepo and D. Pfirsch, Phys. Rev. E **47**, 545 (1993).
[10] D. Correa-Restrepo and D. Pfirsch, Phys. Rev. E **49**, 692 (1994).
[11] D. Correa-Restrepo and D. Pfirsch, in *Proceedings of the 21st EPS Conference on Controlled Fusion and Plasma Physics, Montpellier, 1994*, edited by E. Joffrin, P. Platz, and P. E. Stott (European Physical Society, Geneva, 1994), p. 1398.
[12] D. Correa-Restrepo and D. Pfirsch, Phys. Rev. E **55**, 7449 (1997).
[13] H. Nordman, V. P. Pavlenko, and J. Weiland, Phys. Fluids B **5**, 402 (1993).
[14] D. Pfirsch and D. Correa-Restrepo, Phys. Rev. E **47**, 1947 (1993).
[15] D. Pfirsch, Phys. Rev. E **48**, 1428 (1993).
[16] D. Pfirsch and H. Weitzner, Phys. Rev. E **49**, 3368 (1994).
[17] G. N. Throumoulopoulos and D. Pfirsch, Phys. Rev. E **49**, 3290 (1994).
[18] G. N. Throumoulopoulos and D. Pfirsch, Phys. Rev. E **53**, 2767 (1996).
[19] G. N. Throumoulopoulos and D. Pfirsch, Technical Report No. 6/337, Max-Planck-Institut für Plasmaphysik, Garching bei München, Germany, 1996.
[20] M. Unverzagt, Ph.D. thesis, Max-Planck-Institut für Plasmaphysik, Garching, 1996 (unpublished).
[21] R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983).
[22] H. K. Wimmel, Z. Naturforsch. Teil A **38A**, 601 (1983).
[23] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); Proc. R. Soc. London, Ser. A **246**, 326 (1958); K. Sundermeyer, *Constraint Dynamics*, edited by H. Araki, J. Ehlers, K. Hepp, R. Kippenhahn, H. A. Weidenmüller, and J. Zittartz, Lecture Notes in Physics Vol. 169 (Springer, Berlin, 1982).
[24] D. Correa-Restrepo, D. Pfirsch, and H. K. Wimmel, Physica A **136**, 453 (1986).
[25] D. Pfirsch and D. Correa-Restrepo, Plasma Phys. Controlled Fusion **38**, 71 (1996).
[26] R. J. Groebner, K. H. Burrell, and R. P. Seraydarian, Phys. Rev. Lett. **64**, 3015 (1990).
[27] A. R. Field, G. Fussman, J. V. Hofmann *et al.*, Nucl. Fusion **32**, 1191 (1992).
[28] R. R. Weynants *et al.*, Nucl. Fusion **32**, 837 (1992).
[29] J. B. Taylor, Rev. Mod. Phys. **58**, 741 (1986).